Outline

1. Integration on Manifolds
   - Stokes' Theorem on Manifolds

Integration on Manifolds

The goal of this section is to explain and prove the following

Theorem

Stokes' Theorem on Manifolds. If M is a compact oriented smooth k-dimensional manifold-with-boundary, and ω is a smooth $(k-1)$ form on M, then

$$\int_M d\omega = \int_{\partial M} \omega.$$

We'll do this in three steps:

1. Explain what it means to integrate a k-form over an oriented k-manifold, rather than over a singular k-chain.
2. Check that the orientation of M and the induced orientation of ∂M are "consistent" with our earlier definitions of singular chains and their boundaries.
3. Use partitions of unity to deduce the above version of Stokes' Theorem from the older one on singular chains.

Convention. We will drop the word "smooth" and understand all our manifolds and forms to be smooth of class C^∞.

If ω is a p-form on the k-dimensional manifold-with-boundary M, and if $c : [0,1]^p \to M$ is a singular p-cube, then we define

$$\int_c \omega = \int_{[0,1]^p} c^*\omega,$$

just as we did earlier, and likewise for integrals over p-chains.

When we come to the top dimension k, we are going to require our singular k-cubes $c : [0,1]^k \rightarrow M$ to be rather "nonsingular", in that there be an open set U in $\mathbb{R}^k$ or H^k with $[0,1]^k \subset U$, and a coordinate system $f : U \rightarrow M$ with $f(x) = c(x)$ for all $x \epsilon [0,1]^k$.

If M is oriented, then the singular k-cube c will be called **orientation-preserving** if f is.

Lemma. Let c_1 and $c_2 : [0,1]^k \to M$ be two orientation-preserving singular k-cubes in the oriented k-manifold M, and ω a k-form on M such that $\omega = 0$ outside of $c_1([0,1]^k) \cap c_2([0,1]^k)$. Show that

$$\int_{c_1} \omega = \int_{c_2} \omega.$$

Now let ω be a k-form on the oriented k-manifold M. If there is an orientation-preserving singular k-cube c in M such that $\omega = 0$ outside of $c([0,1]^k)$, then we define

$$\int_M \omega = \int_c \omega.$$

The lemma above shows that this definition does not depend on the choice of c.

Now suppose that ω is an arbitrary (smooth) k-form on M. There is an open cover $\mathcal{O}$ of M such that for each $U \epsilon \mathcal{O}$, there is an orientation-preserving singular k-cube c with $U \subset c([0,1]^k)$. Let Φ be a partition of unity for M subordinate to this cover. We define

$$\int_M \omega = \Sigma_{\phi \epsilon \Phi} \int_M \phi\omega$$

Fact. This definition of $\int_M \omega$ does not depend on the choice of open cover $\mathcal{O}$ or on the partition of unity Φ.

Now let M be a k-dimensional manifold-with-boundary with orientation μ. Let ∂M have the induced orientation $\partial\mu$.

Let c be an orientation-preserving k-cube in M such that the face $c_{(k,0)}$ lies in ∂M, and is the only face of c which has any interior points in ∂M.

Fact. Show that $c_{(k,0)}$ is orientation-preserving if k is even, but not if k is odd.

Thus if ω is a $(k-1)$-form on M which is 0 outside of $c([0,1]^k)$, we have

$$\int c_{(k,0)}\omega = (-1)^k \int_{\partial M} \omega.$$

Furthermore, $c_{(k,0)}$ is the only face of c on which ω is nonzero. Thus

$$\int_{\partial c} \omega = \int_{(-1)^k c_{(k,0)}} \omega = (-1)^k \int_{c_{(k,0)}} \omega = \int_{\partial M} \omega.$$

Now we are ready to prove Stokes' Theorem on manifolds.

Theorem

Stokes' Theorem on Manifolds. *If M is a compact oriented smooth k-dimensional manifold-with-boundary, and ω is a smooth $(k-1)$ form on M, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. Case 1. Suppose there is an orientation-preserving singular k-cube c in the interior of M such that $\omega = 0$ outside of $c([0,1]^k)$. Then

$$\int_c d\omega = \int_{[0,1]^k} c^*(d\omega) = \int_{[0,1]^k} d(c^*\omega) = \int_{\partial([0,1]^k)} c^*\omega = \int_{\partial c} \omega,$$

with the first and last equalities by definition of integration, the second from our definition of d on manifolds, and the third from Stokes' Theorem in Euclidean space.

Case 2. Next suppose there is an orientation-preserving singular k-cube c in M such that $c_{(k,0)}$ is the only face on ∂M, and that $\omega = 0$ outside of $c([0,1]^k)$. Then

$$\int_M d\omega = \int_c d\omega = \int_{\partial c} \omega = \int_{\partial M} \omega,$$

with the first equality following from our definition of integration over M, the second from Stokes' Theorem in Euclidean space, and the last from earlier remarks.

Case 3 - the general case. Take an open cover $\mathcal{O}$ of M and a partition of unity Φ for M subordinate to $\mathcal{O}$ such that for each $\phi \epsilon \Phi$, the form $\phi\omega$ is either as in Case 1 or Case 2. Since M is compact, we can assume that both $\mathcal{O}$ and Φ are finite sets.

Note that $\Sigma_{\phi\epsilon\Phi} d\phi = d(\Sigma_{\phi\epsilon\Phi}\phi) = d(1) = 0$, and so

$$\Sigma_{\phi\epsilon\Phi} d\phi \wedge \omega = 0,$$

hence

$$\Sigma_{\phi\epsilon\Phi} \int_M d\phi \wedge \omega = 0.$$

Therefore

$$\int_M d\omega = \Sigma_{\phi\epsilon\Phi} \int_M \phi d\omega = \Sigma_{\phi\epsilon\Phi} \int_M d\phi \wedge \omega + \phi d\omega$$

$$= \Sigma_{\phi\epsilon\Phi} \int_M d(\phi\omega) = \Sigma_{\phi\epsilon\Phi} \int_{\partial M} \phi\omega = \int_{\partial M} \omega,$$

completing the proof of the Stokes' Theorem on manifolds.